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# Lie algebraic structure for the AKNS system

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**Abstract.** A new approach is proposed to construct the master symmetry and to prove its Lie algebraic structure for the AKNS system.

## 1. Introduction

It is well known that besides isospectral flows $K_n$, the nonlinear evolution equation possesses non-isospectral flows $\sigma_m$ which depend on the space variable explicitly, and $K_n$ and $\sigma_m$ constitute an infinite-dimensional Lie algebra. There are different approaches for deriving the algebraic structure of the evolution equation. In [1-4], by using the recursion operator and its certain algebraic property (which Fuchssteiner calls hereditary [5]), Lie algebraic structures for the Korteweg-de Vries (KdV) system, the Ablowitz-Kaup-Newell-Segur (AKNS) system and the corresponding matrix systems were derived. But for the integrable system not possessing the hereditary recursion operator [6], the above method is not valid. In order to solve this problem, Cheng Yi and Li Yishen have introduced the notion of Lax operators associated with the given spectral problem. According to algebraic properties of these Lax operators, they have also obtained the algebraic structure of the AKNS system [7].

In the present paper we propose a new (elementary, straightforward and pure algebraic) approach for constructing the algebraic structure of the evolution equation. Our basic idea is as follows. We first define implicit expressions of flows $K_n$ and $\sigma_m$ by using the reduction form of Lie group structure equation. Then we give Lie bracket equalities of flows $K_n$ and $\sigma_m$ with the aid of these expressions. Consequently Lie algebraic structure of integrable system can be naturally derived and does not relate to the concept of hereditary symmetries. This appoach is a general one. It can be applied to quite a few integrable systems like the Kadomtsev-Petviashvilli (KP) system. But in this paper, for the sake of simplicity, we will take the AKNS system to illustrate our approach. Furthermore one will find our approach simpler than that mentioned above.

This paper is organized as follows. In section 2 we introduce hierarchies $K_n$ and $\sigma_m$ and their implicit expression for the AKNS system. In section 3 we give the equalities of Lie brackets $[K_n, K_m]$, $[K_n, \sigma_m]$ and $[\sigma_n, \sigma_m]$. Finally we derive the Lie algebraic structure for the AKNS system in section 4.

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## 2. Derivation of the AKNS evolution equation

In order to develop our approach, first we have to derive the well-known AKNS evolution equations [8] in a quite different way.

Consider the Zakharov-Shabat spectral problem

$$\phi_x = M\phi \qquad M = \begin{pmatrix} -\eta & q \\ r & \eta \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{1}$$

where $\phi$ satisfies the time evolution equation

$$\phi_t = N\phi \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2}$$

and $q$, $r$ together with their concerned derivatives vanish rapidly when $x \to -\infty$, while $A$, $B$, $C$ are undetermined functions of $t$, $x$, $\eta$. From the condition $\phi_{tx} = \phi_{xt}$, one gets the Lie group structure equation of spectral problem (1) immediately

$$M_t - N_x + MN - NM = 0. \tag{3}$$

Substituting the matrix expressions of $M$, $N$ into (3) and setting

$$\begin{pmatrix} q \\ r \end{pmatrix} = u \qquad \begin{pmatrix} B \\ C \end{pmatrix} = F \qquad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

we find that

$$A = -I\tilde{u}\delta\gamma F - \eta_t x + A_0 \tag{5}$$

$$u_t = L(\gamma F) - 2\eta\gamma F - 2\gamma(A_0 - \eta_t x)u \tag{6}$$

where $A_0$ is an arbitrary function of $t$, $\eta$, and $\tilde{u}$ is the transposed vector of $u$, while $L$ is an operator defined as follows:

$$L = \gamma(D - 2uI\tilde{u}\delta) \tag{7a}$$

$$D = \frac{\partial}{\partial x} \qquad I = \int_{-\infty}^{x} \mathrm{d}x \qquad DI = ID = 1. \tag{7b}$$

Usually, (6) is called the reduction form of structure equation (3). If we set

$$\eta_t = 0 \qquad A_0 = -2^{n-1}\eta^n \qquad F = F_n = \sum_{j=1}^{n} f_j \eta^{n-j} \tag{8}$$

then comparing the coefficients of the same powers of $\eta$ in (6), it leads to

$$u_t = L(\gamma f_n) \tag{9a}$$

$$\gamma f_{j+1} = \tfrac{1}{2} L(\gamma f_j) \qquad (j = 1, 2, \ldots, n-1) \tag{9b}$$

$$f_1 = 2^{n-1} u. \tag{9c}$$

By induction, we can obtain the isospectral evolution equations

$$u_t = L^n(\gamma u) \qquad (n = 0, 1, 2, \ldots). \tag{10}$$

Setting

$$\eta_t = 2^{m-1}\eta^m \qquad A_0 = 0 \qquad F = G_m = \sum_{j=1}^{m} g_j \eta^{m-j} \tag{11}$$

and comparing the coefficients of the same powers of $\eta$ in (6), we obtain another formula

$$u_t = L(\gamma g_m) \tag{12a}$$

$$\gamma g_{j+1} = \tfrac{1}{2} L(\gamma g_j) \qquad (j = 1, 2, \ldots, m-1) \tag{12b}$$

$$g_1 = 2^{m-1} xu. \tag{12c}$$

Also we find non-isospectral evolution equations by induction,

$$u_t = L^m(\gamma xu) \qquad (m = 0, 1, 2, \ldots). \tag{13}$$

Usually, the right-hand side of (10) and (13) are denoted by $K_n$ and $\sigma_m$, respectively, i.e.

$$K_n = L^n(\gamma u) \qquad \sigma_m = L^m(\gamma xu) \tag{14}$$

and one calls them flows of the AKNS system. From equations (6), (8), (10), (11) and (13), these flows can be written as

$$K_n = L(\gamma F_n) - 2\eta\gamma F_n + (2\eta)^n \gamma u \tag{15a}$$

$$\sigma_m = L(\gamma G_m) - 2\eta\gamma G_m + (2\eta)^m \gamma xu. \tag{15b}$$

We consider the expression (15) as the implicit form for $K_m$ and $\sigma_m$.

Note that setting

$$\eta_t = A_0 = 0 \qquad F = F_n = \sum_{j=0}^{n} f_j \eta^{n-j} \tag{16}$$

and comparing the coefficients of $\eta^{n-j}$ in (6), we easily get $F_n \equiv 0$, so $K_n = 0$ (or $\sigma_m = 0$). Thus, $F_n$ or $G_m$ which satisfy (15$a$) or (15$b$) are unique. This show also that $K_n$, $\sigma_m$ are uniquely determined by (15$a$) and (15$b$) respectively.

## 3. Some Lie bracket equalities

Suppose $f(u)$ is a vector function or operator of the vector $u$ and its derivatives, then

$$f'(u)[\nu] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(u + \varepsilon\nu)|_{\varepsilon=0} \tag{17}$$

is the Gateaux derivative of $f$ in the direction $\nu$. If $g(u)$ is another vector function or operator, it follows easily from the definition

$$(fg)'[\nu] = f'[\nu]g + fg'[\nu]. \tag{18}$$

By convention, the Lie bracket between any two vector functions $f$, $g$ is defined as

$$[f, g] = f'[g] - g'[f]. \tag{19}$$

In terms of expressions (15) and (19), we can get some Lie bracket equalities of $K_n$ and $\sigma_m$.

*Theorem 1.* For $K_n$ given by (15$a$) and

$$K_m = L(\gamma \bar{F}_m) - 2\eta\gamma\bar{F}_m + (2\eta)^m \gamma u \tag{20}$$

$$\theta = F_n'[K_m] - \bar{F}_m'[K_n] + 2\gamma(F_n I\tilde{u}\delta\gamma\bar{F}_m - \bar{F}_m I\tilde{u}\delta\gamma F_n) + (2\eta)^n\gamma\bar{F}_m - (2\eta)^m\gamma F_n \tag{21}$$

we have

$$[K_n, K_m] = L\gamma\theta - 2\eta\gamma\theta. \tag{22}$$

*Theorem 2.* Let $K_n$ and $\sigma_m$ be given by (15*a*) and (15*b*), respectively, and

$$\omega = F_n'[\sigma_m] - G_m'[K_n] + 2\gamma(F_n I\tilde{u}\delta\gamma G_m - G_m I\tilde{u}\delta\gamma F_n) + (2\eta)^n \gamma G_m - (2\eta)^m x\gamma F_n \tag{23}$$

then

$$[K_n, \sigma_m] = L(\gamma\omega) - 2\eta\gamma\omega + (2\eta)^m \gamma F_n. \tag{24}$$

*Theorem 3.* Let

$$\sigma_n = L(\gamma\bar{G}_n) - 2\eta\gamma\bar{G}_n + (2\eta)^n \gamma x u \tag{25}$$

$\sigma_m$ be given by (15*b*), and

$$\zeta = \bar{G}_n'[\sigma_m] - G_m'[\sigma_n] + 2\gamma(\bar{G}_n I\tilde{u}\delta\gamma G_m - G_m I\tilde{u}\delta\gamma\bar{G}_n) + (2\eta)^n \gamma x G_m - (2\eta)^m \gamma x \bar{G}_n \tag{26}$$

then

$$[\sigma_n, \sigma_m] = L(\gamma\zeta) - 2\eta\gamma\zeta + (2\eta)^m \gamma\bar{G}_n - (2\eta)^n \gamma G_m. \tag{27}$$

Here for simplicity, we only prove theorem 2 while others can be proved similarly.

Let $\mathscr{L}$ be a set of matrices of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, $\boldsymbol{\mathscr{L}}$ be a set of all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$, and denote the mapping which maps $\mathscr{L}$ onto $\boldsymbol{\mathscr{L}}$ by $S$, i.e.

$$\begin{pmatrix} a \\ b \end{pmatrix} = S\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}. \tag{28}$$

From (3), (5), (8) and (11), it is easy to see that the implicit form (15) can be written in matrix form (omitting subscript):

$$K_n = S(N_x - MN + NM) \qquad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = -\gamma A + S^{-1}F \tag{29a}$$

$$A = -I\tilde{u}\delta\gamma F - 2^{n-1}\eta^n \tag{29b}$$

$$\sigma_m = S(\bar{N}_x - M\bar{N} + \bar{N}M - 2^{m-1}\eta^m \gamma)$$
$$\bar{N} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & -\bar{A} \end{pmatrix} = -\gamma\bar{A} + S^{-1}G \tag{29c}$$

$$\bar{A} = -I\tilde{u}\delta\gamma G - 2^{m-1}\eta^m x. \tag{29d}$$

Hence we can conclude that

$$[K_n, \sigma_m] = K_n'[\sigma_m] - \sigma_m'[K_n]$$
$$= S(\underline{N}_x - M\underline{N} + \underline{N}M + 2^{m-1}\eta^m(\gamma N - N\gamma)) \tag{30a}$$

$$\underline{N} = N'[\sigma_m] - \bar{N}'[K_n] + N\bar{N} - \bar{N}N \qquad \underline{N} \to 0 (x \to -\infty). \tag{30b}$$

Repeating the procedure from (3) to (7) for this expression (30), and noticing that

$$N\bar{N} - \bar{N}N = \gamma(\bar{B}C - B\bar{C}) + 2S^{-1}\gamma(\bar{A}F - AG) \tag{31a}$$

$$S(\gamma N - N\gamma) = 2\gamma F \tag{31b}$$

we have

$$[K_n, \sigma_m] = L(\gamma\omega) - 2\eta\gamma\omega + (2\eta)^m \gamma F \tag{32a}$$

$$\omega = F'[\sigma_m] - G'[K_n] + 2(F\bar{A} - GA). \tag{32b}$$

Substituting (29*b*) and (29*d*) into (32*b*) we obtain formula (23) immediately.

## 4. Lie algebra of the AKNS system

Consider the AKNS sequence of flows $K_n$, $\sigma_m$ $(n, m = 0, 1, 2, \ldots)$ determined uniquely by (15) and also their linear combination. We denote such a set as $\mathscr{A}$, then $\mathscr{A}$ is obviously an infinite-dimensional linear space. Now let us define the Lie product between elements of $\mathscr{A}$ by Lie bracket (19), then space $\mathscr{A}$ is closed under the Lie product. In fact, we have

$$[K_n, K_m] = 0 \tag{33a}$$

$$[K_n, \sigma_m] = nK_{n+m-1} \tag{33b}$$

$$[\sigma_n, \sigma_m] = (n-m)\sigma_{n+m-1}. \tag{33c}$$

Firstly, because $F_n$, and $\bar{F}_m$ are polynomials in $\eta$, it is clear from the expression (21) that $\theta$ is a polynomial in $\eta$, too. According to formula (22) and the note in the last paragraph of section 2 we obtain $\theta = 0$ immediately, so (33$a$) holds.

Secondly, set

$$\Omega = L(\gamma H) - 2\eta\gamma H + (2\eta)^m \gamma F_n \tag{34}$$

$$F_n = \sum_{j=1}^{n} f_j \eta^{n-j} \tag{35}$$

where $\Omega$ is a vector function which is independent of $\eta$, and $f_j$ satisfy recurrence formulae (9$b$) and (9$c$). If assuming

$$H = \sum_{j=1}^{n+m-1} h_j \eta^{n+m-1-j} \tag{36}$$

and substituting expansions (35) and (36) into (34) and comparing the coefficients of the same powers of $\eta$, we find

$$\Omega = L(\gamma h_{n+m-1}) \tag{37a}$$

$$\gamma h_{j+1} = \tfrac{1}{2} L(\gamma h_j) \qquad (j = n, n+1, \ldots, n+m-2) \tag{37b}$$

$$\gamma h_{i+1} = \tfrac{1}{2} L(\gamma h_i) + 2^{m-1}\gamma f_{i+1} \qquad (i = 1, 2, \ldots, n-1) \tag{37c}$$

$$\gamma h_1 = 2^{m-1}\gamma f_1. \tag{37d}$$

Using the recurrence formulae (37$c$) and (37$d$), we get by induction

$$\gamma h_n = n2^{m-1}\gamma f_n. \tag{38}$$

Also from other recurrence formulae (37$b$) and (38), we obtain by induction

$$\gamma h_{n+m-1} = nL^{m-1}(\gamma f_n) = nK_{n+m-2}. \tag{39}$$

Hence

$$\Omega = nK_{n+m-1} \tag{40}$$

i.e.

$$nK_{n+m-1} = L(\gamma H) - 2\eta\gamma H + (2\eta)^m \gamma F_n. \tag{41}$$

From (41) and (24) we have

$$[K_n, \sigma_m] - nK_{n+m-1} = L\gamma(\omega - H) - 2\eta\gamma(\omega - H). \tag{42}$$

Because $\omega - H$ is a polynomial in $\eta$ (see section 2) it follows that $\omega - H = 0$, so (33$b$) holds.

Finally, we are going to prove formula (33$c$). Let

$$Z = L(\gamma E) - 2\eta\gamma E + (2\eta)^m \gamma \bar{G}_n - (2\eta)^n \gamma G_m \qquad (n > m) \tag{43}$$

where $Z$ is a vector function independent of $\eta$.

Since

$$\bar{G}_n = \sum_{j=1}^{n} \bar{g}_j \eta^{n-j} \qquad \bar{g}_1 = 2^{n-1} xu \tag{44a}$$

$$G_m = \sum_{j=1}^{m} g_j \eta^{m-j} \qquad g_1 = 2^{m-1} xu \tag{44b}$$

where $g_j$, $\bar{g}_j$ satisfy recurrence formula (12$b$), we can take

$$E = \sum_{j=1}^{n+m-1} e_j \eta^{n+m-1-j}. \tag{45}$$

Now substituting (44$a$), (44$b$) and (45) into (43) and comparing the coefficients of same power of $\eta$, we have

$$Z = L(\gamma e_{n+m-1}) \tag{46a}$$

$$\gamma e_{j+1} = \tfrac{1}{2} L(\gamma e_j) \qquad (j = n, n+1, \ldots, n+m-2) \tag{46b}$$

$$\gamma e_{i+1} = \tfrac{1}{2} L(\gamma e_i) + 2^{m-1} \gamma \bar{g}_{i+1} \qquad (i = m, m+1, \ldots, n-1) \tag{46c}$$

$$\gamma e_{l+1} = \tfrac{1}{2} L(\gamma e_l) + \gamma (2^{m-1} \bar{g}_{l+1} - 2^{n-1} g_{l+1}) \qquad (l = 1, 2, \ldots, m-1) \tag{46d}$$

$$e_1 = 0. \tag{46f}$$

It follows from recurrence formulae (46$d$) and (46$f$) by induction that

$$e_l = 0 \qquad (l = 1, 2, \ldots, m-1). \tag{47}$$

Using recurrence formulae (46$c$) and (47), we find by induction

$$\gamma e_n = (n-m) 2^{m-1} \gamma \bar{g}_n. \tag{48}$$

Moreover, from recurrence formulae (46$b$) and (48), we obtained by induction

$$\gamma e_{n+m-1} = (n-m) L^{m-1} (\gamma \bar{g}_n) = (n-m) \sigma_{n+m-2}. \tag{49}$$

Hence

$$Z = (n-m) \sigma_{n+m-1}. \tag{50}$$

From (43) and (27), it follows that

$$[\sigma_n, \sigma_m] - (n-m)\sigma_{n+m-1} = L\gamma(\zeta - E) - 2\eta\gamma(\zeta - E) \tag{51}$$

similarly, we get $\zeta - E = 0$ immediately, so (33$c$) is correct.

From formula (33), we find that the linear space $\mathscr{A}$ is an infinite-dimensional Lie algebra under the Lie product of (19). From this one can construct $\tau$ symmetries of the AKNS evolution equation [2], and $\sigma_n$ are said to be master symmetries [9].

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